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November 7, 1977

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Worst-case Behavior of Local Fix-Point Algorithms

by

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Abstract

We consider methods of locating a completely labelled simplex of a triangulated unit simplex. We show that a class of algorithms which includes the usual pivoting methods may be forced to evaluate a positive fraction of all the vertices of the triangulation. An algorithm which does not have this property is briefly described.

Let T be a triangulation of the unit d-dimensional simplex. If L is a proper labelling of the vertices of T, Sperner's lemma guarantees the existence of a completely labelled d-dimensional simplex. It is well known that the problem of locating a completely labelled simplex is essentially equivalent to the computation of an approximate fixed point.

We consider algorithms A for locating a completely labelled simplex of T. By an algorithm we mean a deterministic procedure which asks for the labels of various of the vertices of T, in such a way that the next vertex for which a label is demanded is determined by the labels already known. Eventually A will locate a completely labelled simplex and halt. Let N(A,T,L) be the number of vertices whose labels are demanded by A when used on (T,L). We are interested in $W(A,T) = \max_L N(A,T,L)$--the most labels A can be forced to evaluate by a malicious labelling L.

In the terminology of [3], we are regarding the labels of T as given by an oracle and W(A,T) is related to the oracle complexity of A. Worst-case analysis of other problems have been treated in [4], [5], and [6].

Most of the popular methods of finding completely labelled simplices proceed by locating a completely labelled (d-1)-dimensional simplex of T on the boundary B of the simplex and then moving through a sequence of almost completely labelled d-dimensional simplices of T. Our definition of a local algorithm is a slight generalization.

Formally, a local algorithm is one that generates a sequence of (d-1) dimensional simplices $S_1,\ldots S_n$ of T and vertices $P_1,\ldots P_n$ such that

(i) (S_i, P_i) is a d-dim simplex of T.

(ii) S_{i+1} is a face of (S_i, p_i) and $p_{i+1} \notin (S_i, p_i)$

(iii) If J is a (d-1)-dimensional simplex in B there is a labelling of B which will make $S_1 = J$.

(iv) If J is a (d-1)-dim simplex of (S_i, p_i) $J \neq S_i$ there is a labelling of p_i which will make $S_{i+1} = J$ and (S_i, p_i) will not be completely labelled.

The crucial property is (iv), which essentially says that the next simplex (S_{i+1}, p_{i+1}) visited by A is determined by the labelling of p_i. The algorithms we have in mind are those in which S_i is a completely labelled (d-1)-dimensional face of (S_i, p_i) and S_{i+1} is the uniquely determined other completely labelled face. It is clear that such algorithms satisfy (iv).

Let $J_2(N)$ be the Whitney-Freudenthal triangulation of the d-dimensional unit simplex (we consider d fixed throughout). See [7, p. 32-35] for a precise description.

Theorem 1: There is a K such that, if A is any local algorithm $W(A, J_2(N)) \geq KN^d$.

Proof: Define a snake* of length M as a sequence $Q_1, Q_2, \ldots Q_M$ of d-dimensional simplices such that (a) Q_i shares a (d-1)-dimensional face with Q_{i+1}. (b) Q_1 has exactly one face on B. (c) the unique point of Q_{i+1} not in Q_i is not in $Q_1, \ldots Q_{i-1}$. (d) the only vertices of Q_i in B are in Q_1.

*This term is taken from [2], where it was used for a related idea in a different context.

We claim $W(A,T) \geq$ the length of the longest snake in T (this is true for any triangulation). By (iii), the boundary B can be labelled so as to make A choose S_1 as $Q_1 \cap B$. Then, for each i, p_i is labelled so as to make $S_{i+1} = Q_i \cap Q_{i+1}$. Condition (d) insures that the propriety of the labelling does not interfere and condition (c) insures $p_i \neq p_j$ for $i \neq j$.

To complete the proof we must construct a long snake in $J_2(N)$. Following [7], we identify $J_2(N)$ with the corresponding triangulation of $U_d(N) = \{(x_1,\ldots x_d) \mid N \geq x_1 \geq \ldots x_d \geq 0\}$. We construct a path P in $U_d(N)$ and obtain the snake by taking Q_1, Q_2,... to be those simplices of $J_2(N)$ such that P passes through their interiors. Intuitively, P will move through the layers of $U_d(N)$ ($N \geq X_1 \geq N-2$, $N-2 \geq X_1 \geq N-4$,...) so that P passes through a positive fraction of all the simplices of $J_2(N)$.

The construction of P takes several steps. Let $0 < f_1 < f_2 < \ldots f_d < f_{d+1} < 1$. For each $1 \leq j \leq d$ and $M \equiv (2j + 1) \bmod 4$ we construct a sequence of points V(j,M) in $U_i(M + 1)$ whose first member is $(M + f_j, M-2 + f_{j-1},\ldots M-2j + 2 + f_1)$ and whose last member is $(2j-1 + f_{j+1}, 2j-3 + f_j,\ldots 1 + f_2)$. Note that V(j,M) is an ordered subset of $U_{j+1})$. We will denote by -V(j,M) the same set in reverse order.

For j=1 V(j,M) is defined as $\langle M + f_1, 1 + f_2\rangle$; For j > 1 V(j,M) is defined as $(M + f_j, V(j-1, M-2))$; $(M + f_j -2, -V(j-1, M-6))$; $(M + f_j - 4, V(j-1, M-6))$;...$(2j + 1 + f_j, V(j-1, j-1))$; $(2j-1 + f_{j+1}, 2j-3 + f_j,\ldots 1 + f_2)$.

Here (x, V(j-1, y)) denotes the sequence of points in $U_j(M+1)$ whose Kth term has first co-ordinate x and 2nd through j-th coordinates determined by the Kth term of V(j-1, y)), and the sequences are concatenated.

Example: For $f_i = .1i$, V(3,11) is (11.3, 9.2, 7.1); (11.3, 9.2, 1.2); (11.3, 7.2, 1.2); (11.3, 7.2, 3.1); (11.3, 5.2, 3.1); (11.3, 5.2, 1.2); (11.3, 3.3, 1.2); (9.3, 3.3, 1.2); (9.3, 5.2, 1.2); (9.3, 5.2, 3.1); (7.3, 5.2, 3.1); (7.3, 5.2, 1.2); (7.3, 3.3, 1.2); (5.4, 3.3, 1.2).

We can now construct P. Let $N-5 \leq M \leq N-2$, $M \equiv 2d + 1 \pmod 4$. P is the polygonal path obtained by connecting the point $(N, M-2+f_{d-1}, M-4+f_{d-2}, \ldots M-2_d+2+f_1)$ to the point $(M+f_d, \ldots M-2d+2+f_1)$ and then connecting the points of V(d,M) in follow-the-dots fashion.

P generates a snake of length $\geq (1/2)^{d+1} \frac{1}{(d-1)!} N^d$ so theorem 1 is proved with $K \geq (1/2)^{d+1} \frac{1}{(d-1)!}$. Q.E.D.

A similar result can be proved for the Kuhn triangulation $K_2(N)$. I conjecture that for any local algorithm A and any triangulation T of mesh $\leq f$ $W(A,T) = 0(f^{-d})$.

However, theorem 1 does not extend to non-local algorithms.

Theorem 2: There exist non-local A such that, for $N=2^L$, $W(A,J_2(N)) \leq KN^{d-1}$.

Proof: A is based on an extension of Sperner's lemma [1, theorem 4]:

Let the triangulated simplex T have each of its vertices labelled with {0,1,...d} in an arbitrary manner (i.e., not necessarily a proper labelling). Then the parity of the number of completely labelled d-dimensional simplices is equal to the parity of the number of (d-1) dimensional simplices in the boundary B which are labelled with {1,...d}.

The propriety of the labelling insures that the unit simplex contains an odd number of (d-1)-dimensional simplices labelled {1,...d} in its

boundary. A divides the unit simplex into two simplices W_1, W_2 whose interiors are disjoint, and such that the hyperplane $W_1 \cap W_2$ does not pass through the interior of any d-dimensional simplex of $J_2(N)$. Either W_1 or W_2 must contain an odd number of (d-1)-simplices labelled {1,...d} in its boundary. The extended Sperner lemma then implies that a completely labelled simplex is in W_1 in one case and W_2 in the other. Then A divides W_i (i=1 or 2) into two simplices W_3 or W_4 in the same manner. Again the extended Sperner lemma tells which of W_3 or W_4 must contain a completely labelled simplex, and so on. At each step, we have a simplex which is known to contain a completely labelled simplex. This simplex is divided into two sub-simplices and the extended Sperner lemma enables one to determine which one must contain a completely labelled simplex. This determination requires evaluation of the labels on the boundaries of the two subsimplices but not in their interiors. After dL bifurcations a completely labelled simplex of $J_2(N)$ is located.

The existence of hyperplanes at each stage which divide the simplex in two and do not pass through the interior of any simplex of the triangulation is a special property of $J_2(2^L)$. As before, we regard $J_2(N)$ as a triangulation of $U_d(N)$. At the first step we divide $U_d(N)$ using the hyperplane $X_1 + X_d = N$. This hyperplane does not pass through the interior of any simplex of $J_2(N)$. Further, the affine mapping $(X_1,...x_d) \rightarrow (N-X_d, N-X_{d-1},...N-X_2, N-X_1)$ carries the half $X_1 + X_d \leq N$ onto the half $X_1 + X_d \geq N$ in such a way that simplices of $J_2(N)$ are mapped to simplicies of $J_2(N)$. This means the two halves are equivalent. After $K < d$ bifurcations one obtains a simplex equivalent to $U_d(N) \cap X_1 + X_{d-k+1} \leq N$. The hyperplane $X_1 + X_{d-k} = N$ divides this simplex into two halves that are

equivalent with respect to the mapping $(X_1,\ldots X_d)\to(N-X_{K-1}, N-X_{K-2},\ldots N-X_1, X_K,\ldots X_d)$. After d bifurcations a simplex equivalent to $U_d(1/2N)$ is obtained. This is divided in half by $X_1 + X_d = \frac{N}{2}$, and so forth.

Each face of $J_2(N)$ contains $\binom{N+d-1}{d-1}$ vertices. Each of the next d divisions requires computing the labels of the vertices on the dividing hyperplane, which is $<\binom{N+d-1}{d-1}$ vertices. The problem is then reduced to $J_2(\frac{N}{2})$ so the total number of vertices for which labels are demanded is $\leq (2d+1)\ \binom{N+d}{d-1}\ (1+(1/2)^{d-1}+ (1/2)^{2d-2} + \ldots$ This establishes $W(A,J_2(N)) \leq (2d+1)\ (1/1-(1/2)^{d-1})\ \binom{N+d-1}{d-1} = O(N^{d-1})$. Q.E.D.

Concluding Remarks:

1. A more detailed analysis shows $W(A,J_2(N)) \leq 3\ \binom{N+d-1}{d-1}$ for N large.
2. For N not a power of 2 $J_2(N)$ can be embedded in $J_2(M)$ for M the next greater power of 2 and a bifurcation algorithm can be performed on $J_2(M)$. This will still require evaluation of $O(M^{d-1})$ labels.
3. The function W is an artificial measure of the amount of time required by an algorithm. It ignores any computational effort other than evaluations of labels, so it is only relevant in situations where individual evaluations of the function are time-consuming. Even in this context, the complete bifurcation algorithm of Theorem 2 is probably impractical for d > 3, but some hybrid of bifurcation and the more familiar local algorithms may be possible. Further discussion may be found in [1], which discusses bifurcation in a non-simplical setting.

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